# THE FUNDAMENTAL GROUP OF AN ALGEBRAIC STACK

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Abstract. We define the notion of fundamental group of an algebraic stack, prove a comparison theorem between the fundamental group of a stack over $\mathbb{C}$ and that of the associated analytic orbifold, show that this notion coincides with that defined with simplicial schemes by [10] and finally prove some short exact sequences involving these groups.

## 1. Pointless fundamental groupoid of a topos

We shall first recall some results from [7].

**Definition 1.1.** A **groupoid** is a category all of whose morphisms are isomorphisms. As a group is a groupoid with a single object, a groupoid may conversely be seen as a "group with several objects".

The **classifying topos** of a groupoid $G$ is the category of presheaves on the opposite category: $\mathfrak{B}G = G^{\mathrm{op}}\mathfrak{Set}$; in particular, the classifying topos of a group $G$ is its category of (left) $G$-sets.

A **progroupoid** is a projective 1-system $(G_i)_{i\in I}$ of groupoids, *i.e.*, a functor $I \longrightarrow \mathfrak{Groupoids}$ from a filtering category $I$; we shall often denote it as "$\varprojlim$" $G_i$.

A **strict progroupoid** is a progroupoid "$\varprojlim$" $G_i$ such that

($i$) the functors $G_i \longrightarrow G_j$ induce bijections between the sets of objects $\operatorname{Ob} G_i \longrightarrow \operatorname{Ob} G_j$;

($ii$) for any objects $X$, $Y$, the functors $G_i \longrightarrow G_j$ induce onto maps $\operatorname{Hom}_{G_i}(X,Y) \longrightarrow \operatorname{Hom}_{G_j}(X,Y)$.

All our progroupoids will be strict.

The classifying topos of a progroupoid "$\varprojlim$" $G_i$ is $\mathfrak{B}$"$\varprojlim$" $G_i$ := $2\text{-}\varprojlim^{\mathfrak{Topos}} \mathfrak{B}G_i$. (The superscript "$\mathfrak{Topos}$" indicates the 2-category in which the 2-limit is taken.)

**Definition 1.2.** Let $\mathscr{T}$ be a topos. An object $X$ is said to be **connected** if whenever $X = X_1 \amalg X_2$, either $X_1$ or $X_2$ is a final object (often denoted $\varnothing$). A **constant object** is a disjoint sum of copies of

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the final object $*$, *i.e.*, it may be written $I_{\mathscr{T}} = \coprod_{i \in I} *$ for some set $I$. The topos $\mathscr{T}$ is said to be connected if its final object is.

An object $F$ is said to be **trivialized by** an object $U$ if its **restriction** $F|_U = (F \times U \longrightarrow U)$ is a constant object of the topos $\mathscr{T}/U$. An object $F$ is said to be locally constant if the objects trivializing it cover the final object,

A **Galois object** in a topos $\mathscr{T}$ is a connected, locally constant object $Y$ such that the morphism $Y \times (\operatorname{Aut} Y)_{\mathscr{T}} \longrightarrow Y \times Y$ is an isomorphism.

A category $\mathfrak{C}$ is said to be **generated by** a set of objects $\mathfrak{K} \subset \operatorname{Ob} \mathfrak{C}$ if for any objects $x$, $y$, for any distinct morphisms $f, g : x \longrightarrow y$, there exists an object $z \in \mathfrak{K}$ and a morphism $k : z \longrightarrow x$ such that $fh \neq gh$.

A **Galois topos** is a topos generated by its Galois objects.

We shall see that Galois toposes are exactly the classifying toposes of strict groupoids.

**Definition 1.3.** A topos is said to be **locally connected** if it is generated by its connected objects; this is equivalent to require that the "constant object" functor $\mathfrak{Set} \longrightarrow \mathscr{T}$ have a left adjoint $\pi$, which we shall call the **connected components functor**.

**Definition 1.4.** A **projective 2-system** in a 2-category $\mathfrak{A}$ is a 2-functor $I \longrightarrow \mathfrak{A}$ from a small filtering 1-category $I$. A projective 2-system of toposes is said to be **strict** if the inverse image functors of the topos morphisms involved are fully faithful.

Let $\mathfrak{A}$ be a 2-category and $\mathfrak{B}$ a (full) sub-2-category. A 1-morphism $f : A \longrightarrow B$ to an object of $\mathfrak{B}$ is said to be **2-universal** if for any morphism $g : A \longrightarrow B'$ to an object of $\mathfrak{B}$, there exists a morphism $h : B \longrightarrow B'$ and a 2-isomorphism $\alpha : hf \longrightarrow g$ such that for any other morphism $h' : B \longrightarrow B'$ and 2-isomorphism $\alpha' : h'f \longrightarrow g$, there exists a *unique* 2-isomorphism $\beta : h \longrightarrow h'$ such that

$$\begin{tikzcd} hf \arrow[d, "\beta f"'] \arrow[dr, "\alpha"] & \\ h'f \arrow[r, "\alpha'"'] & g. \end{tikzcd}$$

**Theorem 1.5.** *Let $\mathscr{T}$ be a locally connected topos.*

(*i*) *The full subcategory* $\operatorname{SLC} \mathscr{T}$ *of disjoint sums of locally constant objects of $\mathscr{T}$ is a Galois topos and the inclusion functor* $\operatorname{SLC} \mathscr{T} \longrightarrow \mathscr{T}$ *is the inverse image functor of a topos morphism* $\mathscr{T} \longrightarrow \operatorname{SLC} \mathscr{T}$ *which is* 2*-universal among morphisms to Galois toposes.*

(*ii*) *The Galois toposes are the strict projective* 2*-limits (in the* 2*-category of toposes) of classifying toposes of groupoids.*

**Remark 1.6.** We shall need the following points of the proof of this theorem.

($i$) If $R$ is a covering sieve of the final object $*$ of $\mathscr{T}$ (we shall write: $R \in J(*)$), denote $\mathrm{LC}(\mathscr{T}, R)$ the full subcategory of $\mathscr{T}$ of the objects trivialized by the connected objects of $R$: it is a topos and the inclusion functor $\mathrm{LC}(\mathscr{T}, R) \longrightarrow \mathscr{T}$ is the inverse image functor of a topos morphism.

($ii$) The toposes $\mathrm{LC}(\mathscr{T}, R)$ are classifying toposes of groupoids; more precisely, if $G$ is the groupoid of points of $\mathrm{LC}(\mathscr{T}, R)$, *i.e.*, the groupoid of topos morphisms $\mathfrak{Set} \longrightarrow \mathrm{LC}(\mathscr{T}, R)$ (and their 2-isomorphisms), then $\mathrm{LC}(\mathscr{T}, R) \approx \mathfrak{B}G$.

($iii$) The Galois topos $\mathrm{SLC}\,\mathscr{T}$ is the projective 2-limit of these toposes:

$$\mathrm{SLC}\,\mathscr{T} \approx \underset{R \in J(*)}{2\text{-}\varprojlim}{}^{\mathfrak{Topos}} LC(\mathscr{T}, R).$$

## 2. Pointed fundamental progroupoid of a topos

The following lemma gives an explicit description of the classifying topos of a progroupoid. In particular, the connected objects of $\mathfrak{B}\text{“}\varprojlim\text{”}\, G_i$ are the (images of the) connected objects of the various $\mathfrak{B}G_i$.

**Lemma 2.1.** *Let $G = \text{“}\varprojlim\text{”} G_i$ be a strict progroup and $\mathfrak{S}_0$ the category whose objects are sets $X$ endowed with an action of one of the $G_i$ (hence of any $G_k$ for $k \longrightarrow i$), connected as presheaves (*i.e.*, they have a single orbit) and whose morphisms are*

$$\mathrm{Hom}\big((X, G_i), (Y, G_j)\big) = \mathrm{Hom}_{G_k}(X, Y) \qquad \textit{for all } k \rightrightarrows \begin{matrix} i \\ j. \end{matrix}$$

*Then, the classifying topos $\mathfrak{B}G$ is equivalent to the category $\mathfrak{S}$ of disjoint sums of objects of $\mathfrak{B}_0$.*

*The same result holds for progroupoids.*

*Proof.* This results from [7, 3.3.6], which states that the projective 2-limit of a *strict* projective system of classifying toposes of groupoids (strict means that the inverse image functors are fully faithful: this is the case here) is the topos of sheaves on $\varinjlim^{\mathfrak{Cat}} \mathfrak{B}G_i$ for the canonical topology and from [7, 2.4.2, 2.4.3] which proves that the category obtained from $\varinjlim^{\mathfrak{Cat}} \mathfrak{B}G_i$ by adding disjoint sums is a topos. □

**Definition 2.2.** A **base point** of a topos $\mathscr{T}$ is a topos morphism $\mathfrak{Set} \longrightarrow \mathrm{SLC}\,\mathscr{T}$.

**Theorem 2.3.** *Let $\mathscr{T}$ be a locally connected topos and $P$ a set of points of $\mathrm{SLC}\,\mathscr{T}$ (*i.e.*, base points of $\mathscr{T}$), at least one in each connected component of $\mathrm{SLC}\,\mathscr{T}$. Let $G_R$ be the groupoid whose objects are elements of $P$ seen as points of $\mathrm{LC}(\mathscr{T}, R)$ and whose morphisms* *are topos 2-isomorphisms* $\mathfrak{Set} \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow}} \mathrm{LC}(\mathscr{T}, R)$ *and let* $\pi_1(\mathscr{T}, P) := $ "$\varprojlim$"$_{R \in J(*)} G_R$. *Then* $\pi_1(\mathscr{T}, P)$ *is a strict progroupoid and* $\mathrm{SLC}\,\mathscr{T} \approx \mathfrak{B}\pi_1(\mathscr{T}, P)$.

*Proof.* To simplify, one may assume that $\mathscr{T}$ is connected. But as the inverse image functors of the morphisms $\mathscr{T} \longrightarrow \mathrm{SLC}\,\mathscr{T}$ or $\mathscr{T} \longrightarrow \mathrm{LC}(\mathscr{T}, R)$ preserve any decomposition $* = X_1 \amalg X_2$ and are fully faithful, $\mathrm{SLC}\,\mathscr{T}$ and $\mathrm{LC}(\mathscr{T}, R)$ are also connected. We may then assume that $P$ is a single point and that the $G_R$ are groups. We have $\mathrm{LC}(\mathscr{T}, R) \approx \mathfrak{B}G_R$, thus $\mathrm{SLC}\,\mathscr{T} \approx \mathfrak{B}\pi_1(\mathscr{T}, P)$.

The projective system $(G_R)_{R \in J(*)}$ is clearly filtering. It remains to prove that the morphisms $f : G_R \longrightarrow G_{R'}$ are surjective: this will come from the fact that a group morphism $f : G' \longrightarrow G$ is onto provided $f^* : \mathfrak{B}G \longrightarrow \mathfrak{B}G'$ is fully faithful. Indeed, as $G$ is a $G$-set, one has then a bijection $G = \mathrm{Aut}_G\, G \longrightarrow \mathrm{Aut}_{G'}\, G = \mathrm{Aut}_{\mathrm{Im}\, f}\, G$. But an $\mathrm{Im}\, f$-automorphism of $G$ is entirely determined by a permutation $\sigma \in \mathfrak{S}_{G/\mathrm{Im}\, f}$ of the orbits and by an automorphism of each orbit; but as the automorphism group of each orbit is isomorphic to $\mathrm{Im}\, f$, one has

$$\mathrm{Aut}_{G'}\, G = \mathrm{Aut}_{\mathrm{Im}\, f}\, G = \left( \prod_{G/\mathrm{Im}\, f} \mathrm{Im}\, f \right) \rtimes \mathfrak{S}_{G/\mathrm{Im}\, f}.$$

But as $G \longrightarrow \mathrm{Aut}_{G'}\, G$ is a bijection, one easily sees that $\mathrm{Im}\, f = G$. □

**Remark 2.4.** By abuse of notation, we shall write $\pi_1 \mathscr{T}$ instead of $\pi_1(\mathscr{T}, P)$ when $P$ is the set of all base points of $\mathscr{T}$.

**Remark 2.5.** All this may be generalized to profinite groupoids in the following way. Let $\mathscr{T}$ be a locally connected topos.

A **finite object** is an object of the form $I_{\mathscr{T}}$ for some finite set $I$, *i.e.*, it is a disjoint sum of a finite number of copies of the final object. A **locally constant finite object** is a locally constant object $F$ such that whenever $U$ trivializes $F$, the restriction $F|_U$ is finite. A **finite Galois object** is a locally constant finite Galois object. A **finite Galois topos** is a topos generated by its finite Galois objects.

The full subcategory $\mathrm{SLCF}\,\mathscr{T}$ of disjoint sums of locally constant finite objects of $\mathscr{T}$ is a finite Galois topos and the inclusion functor $\mathrm{SLCF}\,\mathscr{T} \longrightarrow \mathscr{T}$ is the inverse image functor of a topos morphism; this topos morphism is 2-universal among those to finite Galois toposes. If we let $\mathrm{LCF}(\mathscr{T}, R)$ denote the category of disjoint sums of locally constant finite objects of $\mathscr{T}$ trivialized by the connected elements of a sieve $R$, then $\mathrm{SLCF}\,\mathscr{T} = 2\text{-}\varinjlim_{R \in J(*)} \mathrm{LCF}(\mathscr{T}, R)$.

A **finite groupoid** is a groupoid whose hom-sets are finite. A **profinite groupoid** is a filtered strict projective system of finite groupoids. The finite Galois toposes are the strict projective 2-limits of classifying toposes of finite groupoids.

A **profinite base point** of $\mathscr{T}$ is a topos morphism $\mathfrak{Set} \longrightarrow \mathrm{SLCF}\,\mathscr{T}$. If $P$ is a set of profinite base points of $X$, at least one in each connected component of $\mathrm{SLCF}\,\mathscr{T}$, and if $G_R$ is the groupoid whose objects are elements of $P$ seen as points of $\mathrm{LCF}(\mathscr{T}, R)$ and whose morphisms are topos 2-isomorphisms $\mathfrak{Set} \overset{\Downarrow}{\rightrightarrows} \mathrm{LCF}(\mathscr{T}, R)$, then $\pi_1(\mathscr{T}, P) :=$ “$\varprojlim$”$_{R \in J(*)}$ is a strict finite progroupoid and $\mathrm{SLCF}\,\mathscr{T} \approx \mathfrak{B}\pi_1(\mathscr{T}, P)$.

If $\mathfrak{C}$ is a Galois category and $\mathcal{F} : \mathfrak{C} \longrightarrow \mathfrak{set}$ a fiber functor (see [4]), then the category of disjoint sums of objects of $\mathfrak{C}$ is a connected finite Galois topos and $\mathcal{F}$ induces a profinite base point, still denoted $\mathcal{F}$, then $\widehat{\pi}_1(\mathscr{T}, \{\mathcal{F}\})$ is the profinite fundamental group of $\mathfrak{C}$.

A **quotient groupoid** of a groupoid $G$ is a groupoid morphism $f : G \longrightarrow H$ such that $f$ induces a bijection between the sets of objects of $G$ and $H$ and such that the maps $\mathrm{Hom}_G(x, y) \longrightarrow \mathrm{Hom}(fx, fy)$ be surjective for all $x$, $y \in \mathrm{Ob}\,G$. The **profinite completion** of a groupoid $G$ is the progroupoid $\widehat{G}$ of its finite quotients, the profinite completion of a progroupoid is the progroupoid of the finite quotients of its components. One may check that $\widehat{\pi}_1\mathscr{T}$ is the profinite completion of $\pi_1\mathscr{T}$.

## 3. Fundamental group of a stack

By “stack”, we shall mean Deligne–Mumford stack [2], [13], unless otherwise specified. The following proposition enables us to apply the preceeding theory to define the fundamental group of a stack.

**Proposition 3.1.** *The topos of étale sheaves on an algebraic stack is locally connected.*

*Proof.* The first lemma shows that it is generated by connected schemes, the second that these are connected as sheaves. ☐

**Lemma 3.2.** *Let $\mathscr{F} \rightrightarrows \mathscr{G}$ be two distinct morphisms of étale sheaves on an algebraic stack $X$. Then, there exists a connected scheme $V$, an étale morphism $V \longrightarrow X$ (enabling us to see $V$ as a sheaf on $X$) and a morphism $V \longrightarrow \mathscr{F}$ such that the two morphisms $V \longrightarrow \mathscr{F} \rightrightarrows \mathscr{G}$ be distinct.*

*Proof.* There exists an étale morphism $U \longrightarrow X$ such that the maps $\mathscr{F}U \simeq \mathrm{Hom}(U, \mathscr{F}) \rightrightarrows \mathrm{Hom}(U, \mathscr{G}) \simeq \mathscr{G}U$ be distinct. If $(U_i)_{i \in I}$ are the connected components of $U$, then the maps $\prod \mathrm{Hom}(U_i, \mathscr{F}) \rightrightarrows \prod \mathrm{Hom}(U_i, \mathscr{G})$ are distinct, hence there exists $i \in I$ such that the maps $\mathrm{Hom}(U_i, \mathscr{F}) \rightrightarrows \mathrm{Hom}(U_i, \mathscr{G})$ be distinct. ☐

**Lemma 3.3.** *Let $X$ be a stack and $U \longrightarrow X$ an étale morphism from a connected scheme. Then $U$ is connected as an étale sheaf on $X$.*

*Proof.* Let $U \longrightarrow X$ be an étale morphism from a connected scheme $U$, seen as a presheaf $T \longmapsto \mathrm{Hom}_X(T, U)$ on $X_{\text{ét}}$, which is actually a sheaf. Assume $U = \mathscr{F}_1 \amalg \mathscr{F}_2$ and let us prove that either $\mathscr{F}_1 = \varnothing$ or $\mathscr{F}_2 = \varnothing$.

1. It suffices to prove that for any $Y \longrightarrow X$ in $X_{\text{ét}}$ and for any geometric point $x$ of $Y$, one has $(\mathscr{F})_x = \varnothing$. Indeed, $\{\, p^* : \mathfrak{Sh}\, Y \longrightarrow \mathfrak{Set}$ where $p$ is a geometric point of $Y\,\}$ is a faithful family of fiber functors on $\mathfrak{Sh}\, Y$, thus

$$\{\ \mathfrak{Sh}\, X \xrightarrow{f^*} \mathfrak{Sh}\, Y \xrightarrow{p^*} \mathfrak{Set} \ \text{ where } \ Y \xrightarrow{f} X \ \in X_{\text{ét}} \text{ and } \\ p \text{ is a geometric point of } Y\,\}$$

is a faithful family of fiber functors on $\mathfrak{Sh}\, X$.

2. If $x$ is a geometric point of $U$, then $U_x = *$ and $U_x = (\mathscr{F}_1)_x \amalg (\mathscr{F}_2)_x$, thus $(\mathscr{F}_1)_x = *$ and $(\mathscr{F}_2)_x = \varnothing$ (or the converse).

3. If $x : \operatorname{Spec} k \longrightarrow U$ is a geometric point of $U$ such that $(\mathscr{F}_1)_x = *$ and $(\mathscr{F}_2)_x = \varnothing$, and if $x' : \operatorname{Spec} k' \longrightarrow U$ is a geometric point of $U$ above $x$, *i.e.*,

$$\begin{tikzcd}
\operatorname{Spec} k' \arrow[d] \arrow[dr, "x'"] & \\
\operatorname{Spec} k \arrow[r, "x"'] & U,
\end{tikzcd}$$

then there are maps $U_{x'} \longrightarrow U_x$, $(\mathscr{F}_1)_{x'} \longrightarrow (\mathscr{F}_1)_x$ and $(\mathscr{F}_2)_{x'} \longrightarrow (\mathscr{F}_2)_x$, thus $(\mathscr{F}_2)_{x'} = \varnothing$ and $(\mathscr{F}_1)_x = *$. As a result, the condition "$(\mathscr{F}_1)_x = *$ and $(\mathscr{F}_2)_x = \varnothing$" does not depend on the choice of a geometric point $x$ above a given point $x_0 \in X$.

4. The condition "$(\mathscr{F}_1)_x = *$ and $(\mathscr{F}_2)_x = \varnothing$" is open. Indeed, if $(\mathscr{F}_1)_x \neq \varnothing$, the, as $(\mathscr{F}_1)_x = \varinjlim \mathscr{F}_1(V)$, there exists $V \longrightarrow U$ étale such that $\mathscr{F}_1(V) \neq \varnothing$, and thus $(\mathscr{F}_1)_y \neq \varnothing$ for any $y$ in the image of $V \longrightarrow U$ (which is open). Similarily, the condition "$(\mathscr{F}_2)_x = *$ and $(\mathscr{F}_1)_x = \varnothing$" is open, thus the condition "$(\mathscr{F}_1)_x = *$ and $(\mathscr{F}_2)_x = \varnothing$" is closed. As $U$ is connected, one has $\forall x \in U \quad (\mathscr{F}_1)_x = *$ and $(\mathscr{F}_2)_x = \varnothing$ (or the converse).

5. Let $V \longrightarrow X$ be an object of $X_{\text{ét}}$ and $x : \operatorname{Spec} k \longrightarrow V$ be a geometric point of $V$. In the following diagram, the bold arrow are étale

$$\begin{tikzcd}
& U_x \arrow[rr] \arrow[dl] & & U \underset{X}{\times} V \arrow[dl] \arrow[dr, "p"] & \\
\operatorname{Spec} k \arrow[rr, "x"'] & & V \arrow[dr] & & U \arrow[dl] \\
& & & X. &
\end{tikzcd}$$

One may identify $U_x$ with the set of points $y$ of $U \times_X V$ over $x$.

$$\begin{tikzcd} & & U \underset{X}{\times} V \arrow[dl] \arrow[dr] & \\ \operatorname{Spec} k \arrow[r, "x"'] \arrow[urr, dotted, "y"] & V \arrow[dr] & & U \arrow[dl] \\ & & X & \end{tikzcd}$$

As the decomposition $U = \mathscr{F}_1 \amalg \mathscr{F}_2$ is preserved by pull-backs, $U_x = \coprod_{y \in U_x} U_y$ gives $(\mathscr{F}_2)_x = \coprod_{y \in U_x} (\mathscr{F}_2)_y$. But if $y \in U_x$, one has a map $(\mathscr{F}_2)_y \longrightarrow (\mathscr{F}_2)_{p(y)} = \varnothing$, thus $(\mathscr{F}_2)_y = \varnothing$. □

**Remark 3.4.** This result also holds for the topos of étale sheaves on Artin stacks.

**Remark 3.5.** One may also define the profinite fundamental group of a stack $X$ using étale coverings, *i.e.*, finite étale stack morphisms $Y \longrightarrow X$ instead of locally constant finite sheaves: see [14].

## 4. Comparison theorem

We shall now compare the fundamental group of a stack over $\mathbb{C}$ and that of the associated (generalized) orbifold.

**Definition 4.1.** A **groupoid in a category** $\mathfrak{C}$ (with finite projective limits) is the datum of two objects $G_1$ and $G_0$, and of morphisms $s, t : G_1 \longrightarrow G_0$, $m : G_1 \times_{s, G_0, t} G_1 \longrightarrow G_1$, $e : G_0 \longrightarrow G_1$ and $i : G_1 \longrightarrow G_1$ such that, for any object $T$, $(\operatorname{Hom}(T, G_1), \operatorname{Hom}(T, G_0), s, t, m, e, i)$ be a groupoid, where the maps $s$, $t$, $m$, $e$, $i$ are respectively source, target, composition, identity and inverse. A **topological groupoid** is a groupoid in the category of topological spaces.

An **étale groupoid** is a topological groupoid whose source morphism (and hence, all of whose structure morphisms) is étale. A **proper groupoid** is a topological groupoid such that the morphism $(s, t) : G_1 \longrightarrow G_0 \times G_0$ be proper. One may also consider étale or proper groupoids in the category of schemes.

A **generalized orbifold** is an étale proper groupoid (see [9] for the notion of orbifold).

**Proposition 4.2.** *Let $X$ be a stack of finite type over $\mathbb{C}$ and $R \rightrightarrows U$ a presentation of $X$. The groupoid $R(\mathbb{C}) \rightrightarrows U(\mathbb{C})$ is étale and proper. We call it a (generalized)* **orbifold associated to** *$X$.*

*Proof.* The groupoid is clearly étale and properness results from [6, 3.22]. □

**Definition 4.3.** An **equivariant sheaf** on an étale groupoid $G_1 \rightrightarrows G_0$ is a sheaf $\mathscr{F}$ on $G_0$ and a morphism $\alpha : G_1 {}_s\!\underset{G_0}{\times} \mathscr{F} \longrightarrow \mathscr{F}$ such that

$$\begin{tikzcd} G_1 \underset{G_0}{\times} G_1 \underset{G_0}{\times} \mathscr{F} \arrow[r, "1\times\alpha"] \arrow[d, "m\times 1"'] & G_1 \underset{G_0}{\times} \mathscr{F} \arrow[d, "\alpha"] \\ G_1 \underset{G_0}{\times} \mathscr{F} \arrow[r, "\alpha"'] & \mathscr{F} \end{tikzcd}$$

$$\begin{tikzcd} \mathscr{F} = G_0 \underset{G_0}{\times} \mathscr{F} \arrow[r, "e\times 1"] \arrow[dr, equal] & G_1 \underset{G_0}{\times} \mathscr{F} \arrow[d, "\alpha"] \\ & \mathscr{F} \end{tikzcd} \qquad \begin{tikzcd} G_1 \underset{G_0}{\times} \mathscr{F} \arrow[r, "\alpha"] \arrow[d] & \mathscr{F} \arrow[d] \\ G_1 \arrow[r, "b"'] & G_0. \end{tikzcd}$$

Equivalently, one may require that $(\mathrm{pr}_2, \alpha) : G_1 \times G_0 \mathscr{F} \rightrightarrows \mathscr{F}$ be the source and target maps of a groupoid such that

$$\begin{tikzcd} G_1 \times \mathscr{F} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathscr{F} \arrow[d] \\ G_1 \arrow[r, shift left] \arrow[r, shift right] & G_0 \end{tikzcd}$$

be a groupoid morphism.

The equivariant sheaves and their morphisms form a topos, denoted $\mathfrak{Sh}(G_1 \rightrightarrows G_0)$.

The **fundamental groupoid** of an étale groupoid $G_1 \rightrightarrows G_0$ is defined to be

$$\pi_1(G_1 \rightrightarrows G_0) := \pi_1 \mathfrak{Sh}(G_1 \rightrightarrows G_0).$$

**Remark 4.4.** [13] shows that if $R \rightrightarrows U$ is a presentation of a stack $X$, then $\mathfrak{Sh}\, X \approx \mathfrak{Sh}(R \rightrightarrows U)$.

**Remark 4.5.** As there is an equivalence of categories between the category of coverings of an orbifold and that of étale coverings of one of its presentations, this definition generalizes that of the fundamental group of an orbifold [12, 13.2.5].

**Theorem 4.6.** *Let $X$ be a separated algebraic stack, locally of finite type over $\mathbb{C}$, such that there exists a surjective étale and* finite *morphism $U \longrightarrow X$ from a scheme. We let $R \rightrightarrows U$ be the groupoid defined by this morphism. Then one has*

$$\widehat{\pi}_1 X \approx \widehat{\pi}_1\big(R(\mathbb{C}) \rightrightarrows U(\mathbb{C})\big).$$

*Proof.* We shall identify locally constant finite étale sheaves on the groupoid $R \rightrightarrows U$ with its equivariant étale coverings. It suffices to show that the functor

$$
(1)\qquad \left\{ \begin{array}{ccc} \mathrm{SLCF}\,\mathfrak{Sh}(R \rightrightarrows U) & \longrightarrow & \mathrm{SLCF}\,\mathfrak{Sh}\big(R(\mathbb{C}) \rightrightarrows U(\mathbb{C})\big) \\ \begin{tikzcd} R \underset{U}{\times} F \arrow[r, shift left, "\mathrm{pr}_2"] \arrow[r, shift right, "\alpha"'] \arrow[d] & F \arrow[d, "f"] \\ R \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U \end{tikzcd} & \longmapsto & \begin{tikzcd} R(\mathbb{C}) \underset{U(\mathbb{C})}{\times} F(\mathbb{C}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(\mathbb{C}) \arrow[d, "f"] \\ R(\mathbb{C}) \arrow[r, shift left] \arrow[r, shift right] & U(\mathbb{C}) \end{tikzcd} \end{array}\right.
$$

is an equivalence of categories.

Let $(F, f, \alpha)$ and $(F', f', \alpha')$ be two étale coverings of $R \rightrightarrows U$. Let us first show that the map

$$
(2)\qquad \begin{array}{c} \mathrm{Hom}\big((F, f, \alpha), (F', f', \alpha')\big) \\ \big\downarrow \\ \mathrm{Hom}\big((F(\mathbb{C}), f(\mathbb{C}), \alpha(\mathbb{C})), (F'(\mathbb{C}), f'(\mathbb{C}), \alpha'(\mathbb{C}))\big) \end{array}
$$

is onto. Indeed, any $\varphi_0 : F(\mathbb{C}) \longrightarrow F'(\mathbb{C})$ with $f(\mathbb{C}) = f'(\mathbb{C}) \circ \varphi_0$ is of the form $\varphi_0 = \varphi(\mathbb{C})$ for some $\varphi : F \longrightarrow F'$ with $f = f' \circ \varphi$ (see [4, XII 5.1]); further, as the diagram

$$
\begin{tikzcd} R(\mathbb{C}) \underset{U(\mathbb{C})}{\times} F(\mathbb{C}) \arrow[r, "\alpha"] \arrow[d, "1\times\varphi"'] & F(\mathbb{C}) \arrow[d, "\varphi"] \\ R(\mathbb{C}) \underset{U(\mathbb{C})}{\times} F'(\mathbb{C}) \arrow[r, "\alpha'"'] & F'(\mathbb{C}) \end{tikzcd}
$$

commutes and as that the functor

$$
(3)\qquad \left\{ \begin{array}{ccc} \mathfrak{Sch}/\mathbb{C} & \longrightarrow & \mathfrak{Top} \\ X & \longmapsto & X(\mathbb{C}) \end{array}\right.
$$

is faithful, we see that the diagram

$$
\begin{tikzcd} R \underset{U}{\times} F \arrow[r, "\alpha"] \arrow[d, "1\times\varphi"'] & F \arrow[d, "\varphi"] \\ R \underset{U}{\times} F' \arrow[r, "\alpha'"'] & F' \end{tikzcd}
$$

commutes. The faithfulness of (3) also implies that the map (2) is one-to-one. Hence, the functor (1) is fully faithful.

The functor (1) is essentially surjective. Indeed, if

$$\begin{tikzcd} R(\mathbb{C}) \underset{U(\mathbb{C})}{\times} F(\mathbb{C}) \arrow[r, shift left, "\mathrm{pr}_2"] \arrow[r, shift right, "\alpha_0"'] \arrow[d, "1\times f_0"'] & F(\mathbb{C}) \arrow[d, "f_0"] \\ R(\mathbb{C}) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & U(\mathbb{C}) \end{tikzcd}$$

is an equivariant étale cover of $R(\mathbb{C}) \rightrightarrows U(\mathbb{C})$, then by [4, XII 5.1] there exists an étale cover $f : F \longrightarrow U$ such that $F(C) \simeq F_0$ (we shall thus identify $F(\mathbb{C})$ and $F_0$) and $f(\mathbb{C}) = f_0$. As $U \longrightarrow X$ is finite, so is $(R \times_U F)(\mathbb{C}) \simeq R(\mathbb{C}) \times_{U(\mathbb{C})} F_0 \longrightarrow U(\mathbb{C})$ and thus by [4, XII 5.1] there exists $\alpha : R \times_U F \longrightarrow F$ such that $\alpha(\mathbb{C}) = \alpha_0$. One may then show as above that $(F, f, \alpha)$ is an equivariant covering, again from the fact that $(F_0, f_0, \alpha_0)$ is one and the fact that the functor (3) is faithful. □

## 5. Simplicial schemes

We shall now show that our notion of fundamental groupoid of a stack coincides with that of [10].

**Definition 5.1.** Let $\Delta$ be the **category of simplices**, *i.e.*, the category whose objects are the sets $[n] = \{0, \ldots, n\}$ and whose morphisms are nondecreasing maps, and $\Delta_n$ the full subcategory of $\Delta$ containing $[0], \ldots, [n]$.

Let $\mathscr{T}$ be a locally connected topos and denote by $\pi$ its "connected components" functor.

A **simplicial object** in $\mathscr{T}$ is a functor $\Delta^{\mathrm{op}} \longrightarrow \mathscr{T}$; we shall denote $\Delta^{\mathrm{op}}\mathscr{T}$ the category of simplicial objects of $\mathscr{T}$. The restriction functor $\Delta^{\mathrm{op}}\mathscr{T} \longrightarrow \Delta_n^{\mathrm{op}}\mathscr{T}$ has left and right adjoints, called $n$th **skeleton** and **coskeleton** denoted $\mathrm{Sk}_n$ and $\mathrm{Cosk}_n$.

A **hypercovering** is a simplicial object $X_\bullet$ such that the morphisms

$$X_0 \longrightarrow e$$
$$X_{n+1} \longrightarrow \big(\mathrm{Cosk}_n(X_\bullet)\big)_{n+1}$$

be epic.

If $I$ is a set and $X$ an object of $\mathscr{T}$, set $I \otimes X := \coprod_{i\in I} X$. If $I_\bullet$ is a simplicial set ans $X_\bullet$ a simplicial object, set $I_\bullet \otimes X_\bullet := (n \longmapsto I_n \otimes X_n)$.

A **strict homotopy** between two morphisms $f$, $g : X_\bullet \longrightarrow Y_\bullet$ is a morphism $[1] \otimes X_\bullet \longrightarrow Y_\bullet$ whose restrictions to $[0] \otimes X_\bullet = X_\bullet$ are $f$ and $g$. We call **homotopy** the equivalence relation generated by strict homotopy. We denote $\mathfrak{HR}\mathscr{T}$ the category of hypercoverings of $\mathscr{T}$ and their morphisms up to homotopy. One may show that $(\mathfrak{HR}\mathscr{T})^{\mathrm{op}}$ is filtering [1].

Finally, let $\mathscr{H}$ denote the category of CW-complexes and their morphisms modulo homotopy (which may be seen as a category whose objects are simplicial sets) and pro-$\mathscr{H}$ the category of its proobjects.

**Definition 5.2.** The **homotopy type** of a locally connected topos $\mathscr{T}$ is

$$\{\mathscr{T}\} := \text{“}\varprojlim_{Y_\bullet \in \mathfrak{HR}\mathscr{T}}\text{”}\ \pi Y_\bullet \in \text{pro-}\mathscr{H}.$$

Its **fundamental groupoid** is

$$\pi_1\{\mathscr{T}\} := \text{“}\varprojlim_{Y_\bullet \in \mathfrak{HR}\mathscr{T}}\text{”}\ \pi_1 \pi Y_\bullet$$

(with the same convention as in remark 2.4 for the omission of base points).

**Proposition 5.3.** *If $\mathscr{T}$ is a locally connected topos, then*

$$\mathfrak{B}\pi_1\{\mathscr{T}\} \approx \mathrm{SLC}\,\mathscr{T}.$$

*Proof.* One has

$$\begin{aligned}
\mathfrak{B}\pi_1\{\mathscr{T}\} &= \mathfrak{B}\pi_1\ \text{“}\varprojlim_{U_\bullet \in \mathfrak{HR}\mathscr{T}}\text{”}\ \pi U_\bullet && \text{by the definition of } \{\mathscr{T}\}\\
&= \mathfrak{B}\ \text{“}\varprojlim_{U_\bullet \in \mathfrak{HR}\mathscr{T}}\text{”}\ \pi_1 \pi U_\bullet && \text{by the definition of the fundamental group of a pro-simplicial set}\\
&= \underset{U_\bullet \in \mathfrak{HR}\mathscr{T}}{2\text{-}\varprojlim}\ \mathfrak{B}\pi_1\pi U_\bullet && \text{by the definition of the classifying space of a progroup}\\
&\approx \underset{U_\bullet \in \mathfrak{HR}\mathscr{T}}{2\text{-}\varprojlim}\ \mathrm{LC}(\mathscr{T}, U_0) && \text{from [1, 10.6]}\\
&\approx \underset{R \in J(*)}{2\text{-}\varprojlim}\ \mathrm{LC}(\mathscr{T}, R).
\end{aligned}$$

To prove the last equivalence, it suffices to show that any sieve $R$ is finer than a sieve generated by a single object $U_0$. One may be tempted to set $U_0 = \coprod_{(U \to *) \in R} U$, but this coproduct need not be small. Instead, as there exists a *set* $\mathfrak{K} \subset \mathrm{Ob}\,\mathscr{T}$, generating $\mathscr{T}$, one may set

$$U_0 = \coprod_{\substack{C \in \mathfrak{K} \text{ such that} \\ C \to * \text{ factors} \\ \text{through a } (U \to *) \in R}} C.$$

□

## 6. Short exact sequences

**Theorem 6.1.** *Let $\mathscr{T}$ be a connected locally connected topos, $x$ a point of* $\mathrm{SLC}\,\mathscr{T}$ *and* $\mathcal{F} : \mathrm{SLC}\,\mathscr{T} \longrightarrow \mathfrak{Set}$ *the corresponding fiber functor. Let $Y$ be a Galois object of* $\mathrm{SLC}\,\mathscr{T}$.

(*i*) *The category* $\mathrm{SLC}\,\mathscr{T}/Y$ *is a Galois topos and a point* $a \in \mathcal{F}(Y)$ *defines a fiber functor* $\mathrm{SLC}\,\mathscr{T} \longrightarrow \mathfrak{Set}$.

(*ii*) *The functor*

$$\begin{cases} \mathrm{SLC}\,\mathscr{T} & \longrightarrow & (\mathrm{SLC}\,\mathscr{T})/Y \\ X & \longmapsto & (X \times Y \longrightarrow Y) \end{cases}$$

*induces a topos morphism* $(\mathrm{SLC}\,\mathscr{T})/Y \longrightarrow \mathrm{SLC}\,\mathscr{T}$.

(*iii*) *There is a short exact sequence*

$$1 \longrightarrow \pi_1(Y,a) \longrightarrow \pi_1(\mathscr{T},x) \longrightarrow \operatorname{Aut} Y \longrightarrow 1.$$

*Proof.* One may assume that $\mathscr{T}$ is the classifying topos of a progroup and that $\mathcal{F}$ is the forgetful functor $G\text{-}\mathfrak{Set} \longrightarrow \mathfrak{Set}$. The following lemma identifies $G\text{-}\mathfrak{Set}/Y$ as the classifying topos of a subprogroupoid $H$ of $G$, thereby proving $(i)$ and $(ii)$.

**Lemma 6.2.** *Let $G$ be a progroup, $Y$ a connected $G$-set, $a \in Y$ and $H$ the stibilizer of $a$. Then, there is an equivalence of categories*

$$G\text{-}\mathfrak{Set}/Y \approx H\text{-}\mathfrak{Set}.$$

*Proof.* Consider the functor

$$\begin{cases} G\text{-}\mathfrak{Set}/Y & \longrightarrow & H\text{-}\mathfrak{Set} \\ (\, Z \xrightarrow{\pi} Y \,) & \longmapsto & \pi^{-1}(\{a\}). \end{cases}$$

Let $Z$ and $Z'$ be objects of $G\text{-}\mathfrak{Set}/Y$. The map

$$\operatorname{Hom}_{Y,G}(Z,Z') \longrightarrow \operatorname{Hom}_H\Big(\pi^{-1}\big(\{a\}\big), \pi'^{-1}\big(\{a\}\big)\Big)$$

is injective, because a morphism $f : Z \longrightarrow Z'$ is entirely determined by its fiber $f_a$. Indeed, as $Y$ is connected, for any $z \in Z$, there exists $g \in G$ such that $\pi(z) = g \cdot a$, thus $f(z) = f(gg^{-1} \cdot z) = g \cdot f(g^{-1} \cdot z))$, for $g^{-1} \cdot z \in \pi^{-1}(\{a\})$.

To show that the map is onto, let $f_a : \pi^{-1}(\{a\}) \longrightarrow \pi'^{-1}(\{a\})$ be an $H$-morphisms and set $f(z) = g \cdot f_a(\gamma^{-1} \cdot z)$ if $\pi(z) = g \cdot a$. This is well-defined, for if $\pi(z) = g \cdot a = g' \cdot a$, then $g^{-1}g'a = a$ thus $h = g^{-1}g' \in H$, but as $f_a$ is $H$-equivariant, $h^{-1}f_a(g^{-1} \cdot z) = f_a(h^{-1}g^{-1}z)$, thus $gf_a(g^{-1}z) = g'f_a(g'^{-1}z)$. Further, the map $f$ is equivariant: $g' \cdot f(z) = g'g \cdot f_a((g'g)^{-1} \cdot g'z) = f(g' \cdot z)$.

Our functor is thus fully faithful. As for essential surjectivity, let $U$ be an $H$-set and define a $G$-action on $Y \times U$ as follows: for all $y \in Y$, choose $\gamma_y \in G$ such that $y = \gamma_y \cdot a$ (one may set $\gamma_a = 1$) and set $g \cdot (y,u) = (g \cdot y, \gamma_{gy}^{-1} g \gamma_y u)$. This is clearly a $G$-action and the action of $H$ on the fiber is the desired one. □

The following lemma shows that $\pi_1(Y,a)$ is a normal subgroup of $\pi_1(\mathscr{T},x)$ and identifies the quotient, thereby establishing $(iii)$. □

**Lemma 6.3.** *Let $G$ be a progroup, $Y$ a Galois $G$-set, $a \in Y$ and $H$ the stabilizer of $a$. Then, the subprogroup $H$ is normal and there is an isomorphism*

$$G/H \simeq \operatorname{Aut} Y.$$

*Proof.* Let us first assume that $G$ is a group. As $Y$ is Galois, the map

$$\Psi : \begin{cases} Y \times \operatorname{Aut} Y & \longrightarrow & Y \times Y \\ (y,\varphi) & \longmapsto & (y,\varphi(y)) \end{cases}$$

is bijective so we may define

$$\Phi : \begin{cases} G & \longrightarrow & \operatorname{Aut} Y \\ g & \longmapsto & \varphi \text{ such that } \Psi(a,\varphi) = (a, g\cdot a). \end{cases}$$

This map is onto: as $Y$ is connected, an automorphism $\varphi \in \operatorname{Aut} Y$ is entirely determined by $\varphi(a)$, but there exists a $g \in G$ such that $\varphi(a) = g \cdot a$, thus $\varphi = \Phi(g)$. This map is clearly a group morphism and its kernel is readily seen to be $H$.

In the case $G =$ " $\varprojlim$" $G_i$ is a progroup, one has short exact sequences

$$1 \longrightarrow H_i \longrightarrow G_i \longrightarrow \operatorname{Aut} Y \longrightarrow 1$$

thus, by [5, I.8.9.2],

$$1 \longrightarrow H \longrightarrow G \longrightarrow \operatorname{Aut} Y \longrightarrow 1.$$

□

**Corollary 6.4.** *Let $X$ be a connected scheme, $G$ a finite group actiong on $X$ and $x$ a geomertic point of $X$. Then, there is a short exact sequence*

$$1 \longrightarrow \widehat{\pi}_1(X,x) \longrightarrow \widehat{\pi}_1([X/G],x) \longrightarrow G \longrightarrow 1.$$

*Proof.* Let us apply the theorem to $\mathscr{T} = \operatorname{SLCF} \mathfrak{Sh}(G\times X \rightrightarrows X)$, the topos of disjoint sums of $G$-equivariant étale coverings of $X$ and the object $Y = G \times X$, where the action of $G$ and the morphism $Y \longrightarrow X$ are

$$\begin{cases} G\times Y & \longrightarrow & Y \\ (h,g,x) & \longmapsto & (gh^{-1}, hx) \end{cases} \qquad \pi : \begin{cases} Y & \longrightarrow & X \\ (g,x) & \longmapsto & g\cdot x \end{cases}$$

To prove that $G = \operatorname{Aut} Y$, remark that the one-to-one morphism

$$\Phi : \begin{cases} G & \longrightarrow & \operatorname{Aut} Y \\ h & \longmapsto & \Phi(h) : \begin{cases} Y & \longrightarrow & Y \\ (g,x) & \longmapsto & (h^{-1}g, h\cdot x) \end{cases} \end{cases}$$

is onto: indeed, if $\varphi \in \operatorname{Aut} Y$, $x \in X$, as $\varphi$ is an automorphism over $X$, one may write $\varphi(1,x) = (h^{-1}, h\cdot x)$ with $h \in G$: thus $\varphi$ and $\Phi(h)$ coincide at a point $(1,x)$ of $Y$, hence on the connected component $\{1\}\times X$, hence on its orbit $Y$.

To show that $Y$ is Galois, it suffices to remark that

$$Y \underset{X}{\times} (\operatorname{Aut} X) = \{(g,h,x,y) \in G\times X\times G\times X \ : \ g\cdot x = y\}$$

$$Y \underset{X}{\times} Y = \{(g,h,x,y) \in G\times X\times G\times X \ : \ g\cdot x = h\cdot y\}.$$

Finally, as $\mathscr{T}/Y \approx \operatorname{SLCF} \mathfrak{Sh}\, X$, the short exact sequence of the theorem

$$1 \longrightarrow \widehat{\pi}_1 \mathscr{T}/Y \longrightarrow \widehat{\pi}_1 \mathscr{T} \longrightarrow G \longrightarrow 1$$

reads

$$1 \longrightarrow \widehat{\pi}_1 X \longrightarrow \widehat{\pi}_1 [X/G] \longrightarrow G \longrightarrow 1.$$

□

**Remark 6.5.** The same result holds for stacks quotiented by a finite group.

**Corollary 6.6.** *Let $X$ be a connected quasi-compact algebraic stack on a field $k$, let $\bar{k}$ be a separable closure of $k$ and let $x$ be a point of $\bar{X} = X \otimes_k \bar{k}$. Then, one has a short exact sequence of profinite fundamental groups*

$$1 \longrightarrow \widehat{\pi}_1(\bar{X}, x) \longrightarrow \widehat{\pi}_1(X, x) \longrightarrow \mathrm{Gal}(\bar{k} : k) \longrightarrow 1.$$

*Proof.* By lemma 6.7, for each Galois extension $k' : k$, $X \otimes_k k'$ is a Galois object in SLCF $\mathfrak{Sh}\, X$ and the short exact sequence of the theorem reads

$$1 \longrightarrow \widehat{\pi}_1(X \otimes_k k') \longrightarrow \widehat{\pi}_1(X) \longrightarrow \mathrm{Gal}(k' : k) \longrightarrow 1,$$

thus, from [5, I.8.9.2],

$$1 \longrightarrow \varprojlim_{k':k} \widehat{\pi}_1(X \otimes_k k') \longrightarrow \varprojlim_{k':k} \widehat{\pi}_1(X) \longrightarrow \varprojlim_{k':k} \mathrm{Gal}(k' : k) \longrightarrow 1,$$

but lemma 6.8 identifies the first term, yielding

$$1 \longrightarrow \widehat{\pi}_1(X \otimes_k k') \longrightarrow \widehat{\pi}_1(X) \longrightarrow \mathrm{Gal}(\bar{k} : k) \longrightarrow 1$$

as desired. □

**Lemma 6.7.** *Let $X$ be a connected algebraic stack over a field $k$. Then $X \underset{k}{\otimes} k' \longrightarrow X$ is Galois. and* $\mathrm{Gal}(k' : k) \simeq \mathrm{Aut}_X(X \otimes_k k')$

*Proof.* Let $s, t : R \rightrightarrows U$ be a presentation of $X$, $\operatorname{Spec} A_i$ affine open sets covering $U$, $\operatorname{Spec} B_{ijk}$ affine open sets covering $s^{-1} \operatorname{Spec} A_i \cap t^{-1} \operatorname{Spec} A_j$. An automorphism of $X \otimes_k k'$ over $X$ induces automorphisms of $\operatorname{Spec} k' \otimes_k B_{ijk}$ over $\operatorname{Spec} B_{ijk}$ and of $\operatorname{Spec} k' \otimes_k A_i$ over $\operatorname{Spec} A_i$, compatible with $s$ and $t$; each of these uniquely determines an automorphism of $k'$ over $k$, but as the groupoid $R \rightrightarrows U$ is connected, it is always the same automorphism of $k' : k$.

To show that $X'$ is Galois, just compute:

$$\begin{aligned}
X' \underset{X}{\times} X' &= \left( X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} k' \right) \underset{X}{\times} \left( X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} k' \right) \\
&= X \underset{\operatorname{Spec} k}{\times} \left( \operatorname{Spec} k' \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} k' \right) \\
&= X \underset{\operatorname{Spec} k}{\times} (\operatorname{Spec} k' \times \mathrm{Gal}(k' : k)) && \text{for } k' : k \text{ is Galois} \\
&= X \underset{\operatorname{Spec} k}{\times} \mathrm{Aut}_X(X \otimes_k k').
\end{aligned}$$

□

**Lemma 6.8.** *Let $X$ be a connected algebraic stack over a field $k$, with a quasi-compact presentation, and let $\bar{k}$ be a separable closure of $k$. Then, there is an isomorphism of profinite groups*

$$\widehat{\pi}_1(X \otimes_k \bar{k}) \simeq \underset{\substack{k':k \text{ finite} \\ \text{Galois extension}}}{\text{“}\varprojlim\text{”}} \widehat{\pi}_1(X \otimes_k k').$$

*Proof.* It suffices to prove that any étale covering of $\bar{X} = X \otimes_k \bar{k}$ comes from a covering of $X' = X \otimes_k k'$ for some *finite* Galois extension $k' : k$. Let $s, t : R \rightrightarrows U$ be a presentation of $X$, with $R$ and $U$ quasi-compact, let $\alpha : R \times_{\bar{U}} F \longrightarrow F$ be an equivariant étale cover of $\bar{R} \rightrightarrows \bar{U}$, let $\operatorname{Spec} A_i$ be a finite number of open affine sets covering $U$, let $\operatorname{Spec} B_{ij}$ be a finite number of open affine sets covering $\alpha^{-1} \operatorname{Spec} A_i$. The morphisms $F \longrightarrow \bar{U}$ and $\alpha$ may be written, locally,

$$\operatorname{Spec} \frac{A_i \otimes \bar{k}[x_1, \ldots, x_{n_i}]}{(f_{i,1}, \ldots, f_{i,n_i})} \longrightarrow \operatorname{Spec} A_i \otimes \bar{k},$$

$$\operatorname{Spec} B_{ij} \underset{A_i}{\otimes} \frac{A_i \otimes \bar{k}[x_1, \ldots, x_{n_i}]}{(f_{i,1}, \ldots, f_{i,n_i})} \longrightarrow \operatorname{Spec} \frac{A_i \otimes \bar{k}[x_1, \ldots, x_{n_i}]}{(f_{i,1}, \ldots, f_{i,n_i})}$$

and only involve a finite number of elements of $\bar{k}$: thus, $F$ is defined over the finite Galois extension generated by these elements. □

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